Deviation from Maxwell distribution in granular gases with constant restitution coefficient

Nikolai V. Brilliantov[1,2] and Thorsten Pöschel[1,*]
[1]Institut für Physik, Humboldt-Universität zu Berlin, Invalidenstraße 110, D-10115 Berlin, Germany
[2]Physics Department, Moscow State University, Moscow 119899, Russia

(Received 28 June 1999)

We analyze the velocity distribution function of force-free granular gases in the regime of homogeneous cooling when deviations from the Maxwellian distribution may be accounted only by the leading term in the Sonine polynomial expansion, quantified by the second coefficient a_2. We go beyond the linear approximation for a_2 and find three different values (three roots) for this coefficient which correspond to a scaling solution of the Boltzmann equation. The stability analysis performed showed, however, that among these three roots only one corresponds to a stable scaling solution. This is very close to a_2, obtained in previous studies in a linear with respect to a_2 approximation.

PACS number(s): 81.05.Rm, 36.40.Sx, 51.20.+d, 66.30.Hs

Granular gases, i.e., rarefied systems composed of inelastically colliding particles have been of particular interest during the last decade (e.g., Refs. [1–4]). Compared to gases of elastically colliding particles, the dissipation of energy at inelastic collisions leads to some novel phenomena in these systems such as clustering (e.g., Ref. [1]), formation of vortex patterns (e.g., Ref. [2]), etc. Before clustering starts, the granular gas being initially homogeneous, keeps for some time its homogeneity, although its temperature permanently decreases. This regime is called the homogeneous cooling regime (HC).

In the present study we address the properties of the velocity distribution of granular particles in the regime of HC, such as the deviation from the Maxwellian distribution and the stability of the distribution function. We assume that the restitution coefficient ϵ does not depend on the impact velocity, i.e., that ϵ=const. The properties of the velocity distribution for the system with impact-velocity dependent coefficient of restitution (e.g., Ref. [5]) will be addressed elsewhere [6].

It is well known that granular gases in the HC regime do not reveal Maxwellian distribution (e.g., Refs. [3,4,7,8]). The high-velocity tail is overpopulated [3,8], while the main part of the distribution is described by the sum of the Maxwellian and correction to it, written in terms of the Sonine polynomial expansion (e.g., Refs. [3,4,7]). Usually only the leading, second term, in this expansion is taken into account [3,4,7], moreover in previous studies [3,4] only linear analysis with respect to the coefficient a_2, which refers to this second term has been performed. Finding that a_2, obtained within the linear approximation, is small, the authors of Refs. [4,3] conclude *a posteriori* that the linear approximation is valid.

In our approach we also assume that one can restrict oneself to the leading term in the Sonine polynomial expansion. However, we go beyond the linear approximation with respect to the coefficient a_2 and perform complete analysis within this level of the system description. We found three different values of a_2 which correspond to the scaling solution of the Boltzmann equation. The stability analysis for the velocity distribution function shows, however, that only one value of a_2 corresponds to a physically acceptable stable scaling solution. The stable solution is close to the result previously obtained within the linear analysis [3].

To introduce notations and specify the problem we briefly sketch the derivation of the coefficient a_2 [3,4]. We introduce the (time-dependent) temperature $T(t)$, and the thermal velocity $v_0(t)$, which are related to the velocity distribution function $f(\mathbf{v},t)$ for 3D systems as

$$\frac{3}{2}nT(t)=\int d\mathbf{v}\,\frac{v^2}{2}f(\mathbf{v},t)=\frac{3}{2}nv_0^2(t). \tag{1}$$

Here n is the number density and the particles are assumed to be of unit mass $(m=1)$. The inelasticity of collisions is characterized by the coefficient of normal restitution ϵ, which relates the after-collisional velocities $\mathbf{v}_1^*$, $\mathbf{v}_2^*$ to the precollisional ones, $\mathbf{v}_1$, $\mathbf{v}_2$ as

$$\mathbf{v}_{1/2}^*=\mathbf{v}_{1/2}\mp\frac{1}{2}(1+\epsilon)(\mathbf{v}_{12}\cdot\mathbf{e})\mathbf{e}, \tag{2}$$

where $\mathbf{v}_{12}=\mathbf{v}_1-\mathbf{v}_2$ is the relative velocity, the unit vector $\mathbf{e}=\mathbf{r}_{12}/|\mathbf{r}_{12}|$ gives the direction of the vector $\mathbf{r}_{12}=\mathbf{r}_1-\mathbf{r}_2$ at the instant of the collision. The time-evolution of the velocity distribution function is subjected to the Enskog-Boltzmann equation, which for the force-free case reads [3,9]

$$\frac{\partial}{\partial t}f(\mathbf{v},t)=g_2(\sigma)\sigma^2\int d\mathbf{v}_2\int d\mathbf{e}\,\Theta(-\mathbf{v}_{12}\cdot\mathbf{e})|\mathbf{v}_{12}\cdot\mathbf{e}|\times\left\{\frac{1}{\epsilon^2}f(\mathbf{v}_1^{**},t)f(\mathbf{v}_2^{**},t)-f(\mathbf{v}_1,t)f(\mathbf{v}_2,t)\right\}, \tag{3}$$

where σ is the diameter of particles, $g_2(\sigma)=(2-\eta)/2(1-\eta)^3$ ($\eta=\frac{1}{6}\pi n\sigma^3$ is the packing fraction) denotes the contact value of the two-particle correlation function [10], which accounts for the increasing collision frequency due to the excluded volume effects; $\Theta(x)$ is the Heaviside function. The velocities $\mathbf{v}_1^{**}$ and $\mathbf{v}_2^{**}$ refer to the pre-collisional ve-

*URL: http://summa.physik.hu-berlin.de/~thorsten/

1063-651X/2000/61(3)/2809(4)/$15.00

locities of the so-called inverse collision, which results with $\mathbf{v}_1$ and $\mathbf{v}_2$ as the after-collisional velocities. The factor $1/\epsilon^2$ in the gain term appears respectively from the Jacobian of the transformation $d\mathbf{v}_1^{**}d\mathbf{v}_2^{**}\to d\mathbf{v}_1 d\mathbf{v}_2$ and from the relation between the lengths of the collisional cylinders $\epsilon|\mathbf{v}_{12}^{**}\cdot\mathbf{e}|dt=|\mathbf{v}_{12}\cdot\mathbf{e}|dt$ [3,9].

Assuming that the velocity distribution function is of a scaling form

$$f(\mathbf{v},t)=\frac{n}{v_0^3(t)}\tilde{f}(\mathbf{c}) \tag{4}$$

one can show that the scaling function satisfies the *time-independent* equation [3]

$$\frac{\mu_2}{3}\left(3+c_1\frac{\partial}{\partial c_1}\right)\tilde{f}(\mathbf{c})=\tilde{I}(\tilde{f},\tilde{f}) \tag{5}$$

with the dimensionless collision integral

$$\tilde{I}(\tilde{f},\tilde{f})=\int d\mathbf{c}_2\int d\mathbf{e}\Theta(-\mathbf{c}_{12}\cdot\mathbf{e})|\mathbf{c}_{12}\cdot\mathbf{e}|\{\epsilon^{-2}\tilde{f}(\mathbf{c}_1^{**})\tilde{f}(\mathbf{c}_2^{**})-\tilde{f}(\mathbf{c}_1)\tilde{f}(\mathbf{c}_2)\} \tag{6}$$

and with its moments [3]

$$\mu_p\equiv-\int d\mathbf{c}_1 c_1^p\tilde{I}(\tilde{f},\tilde{f}), \tag{7}$$

while the time-evolution of temperature reads

$$dT/dt=-(2/3)BT\mu_2, \tag{8}$$

where $B=B(t)\equiv v_0(t)g_2(\sigma)\sigma^2 n$.

To proceed we use the Sonine polynomial expansion for the velocity distribution function [3,4]

$$\tilde{f}(\mathbf{c})=\phi(c)\left\{1+\sum_{p=1}^{\infty}a_pS_p(c^2)\right\}, \tag{9}$$

where $\phi(c)\equiv\pi^{-d/2}\exp(-c^2)$ is the Maxwellian distribution and the first few Sonine polynomials read $S_0(x)=1$, $S_1(x)=-x^2+\frac{3}{2}$, $S_2(x)=x^2/2-5x/2+\frac{15}{8}$, etc. Multiplying both sides of Eq. (5) with c_1^p and integrating by parts over $d\mathbf{c}_1$, we obtain [3]

$$\frac{\mu_2}{3}p\langle c^p\rangle=\mu_p, \tag{10}$$

where we define

$$\langle c^p\rangle\equiv\int c^p\tilde{f}(\mathbf{c},t)d\mathbf{c}. \tag{11}$$

The odd moments $\langle c^{2n+1}\rangle$ are zero, while the even ones, $\langle c^{2n}\rangle$, may be expressed in terms of a_k with $0\leqslant k\leqslant n$. Calculations show that $\langle c^2\rangle=\frac{3}{2}$, implying $a_1=0$, according to the definition of the temperature (1) (e.g., Ref. [3]), and that $\langle c^4\rangle=\frac{15}{4}(1+a_2)$.

Now we assume, that the dissipation is not large, so that the deviation from the Maxwellian distribution may be accurately described only by the second term in the expansion (9) with all high-order terms with $p>2$ discarded. Then Eq. (10) is an equation for the coefficient a_2. Using the above results for $\langle c^2\rangle$ and $\langle c^4\rangle$ it is easy to show that Eq. (10) converts for $p=2$ into identity, while for $p=4$ it reads

$$5\mu_2(1+a_2)-\mu_4=0. \tag{12}$$

The coefficients μ_p may be expressed in terms of a_2 due to the definition (7) and the assumption $\tilde{f}=\phi(c)[1+a_2S_2(c^2)]$. Using the properties of the collision integral one obtains for μ_p [3]

$$\mu_p=-\frac{1}{2}\int d\mathbf{c}_1\int d\mathbf{c}_2\int d\mathbf{e}\Theta(-\mathbf{c}_{12}\cdot\mathbf{e})|\mathbf{c}_{12}\cdot\mathbf{e}|\phi(c_1)\phi(c_2)\{1+a_2[S_2(c_1^2)+S_2(c_2^2)]+a_2^2S_2(c_1^2)S_2(c_2^2)\}\Delta(c_1^p+c_2^p),$$

where $\Delta\psi(\mathbf{c}_i)\equiv[\psi(\mathbf{c}_i^*)-\psi(\mathbf{c}_i)]$ denotes change of some function $\psi(\mathbf{c}_i)$ in a direct collision. Calculations, similar to that, described in Ref. [3], yield (details are given in Ref. [6]):

$$\mu_2=\sqrt{2\pi}(1-\epsilon^2)\left(1+\frac{3}{16}a_2+\frac{9}{1024}a_2^2\right), \tag{13}$$

$$\mu_4=4\sqrt{2\pi}\{T_1+a_2T_2+a_2^2T_3\}, \tag{14}$$

with

$$T_1=\frac{1}{4}(1-\epsilon^2)\left(\frac{9}{2}+\epsilon^2\right), \tag{15}$$

$$T_2=\frac{3}{128}(1-\epsilon^2)(69+10\epsilon^2)+\frac{1}{2}(1+\epsilon),$$

$$T_3=\frac{1}{64}(1+\epsilon)+\frac{1}{8192}(1-\epsilon^2)(9-30\epsilon^2).$$

The coefficients μ_2 and μ_4 were provided in Ref. [3] up to terms of the order of $\mathcal{O}(a_2)$. One obtains the coefficient a_2 in the Sonine polynomial expansion in this approximation by substituting Eqs. (13), (14) into Eq. (12) and discarding in Eqs. (13), (14) all terms of the order of $\mathcal{O}(a_2^2)$:

$$a_2^{\mathrm{NE}}=\frac{16(1-\epsilon)(1-2\epsilon^2)}{81-17\epsilon+30\epsilon^2(1-\epsilon)}. \tag{16}$$

Calculations including the next order terms $\mathcal{O}(a_2^2)$ in the coefficients μ_2 and μ_4 show that Eq. (12) is a cubic equation, which for physical values of ϵ, $0\leqslant\epsilon\leqslant 1$, has three different real roots, as it shown in Fig. 1.

Although the cubic equation may be generally solved, the resultant expressions for the roots are too cumbersome to be written explicitly. However, one of the roots (the middle one) is rather small and close to that given by Eq. (16), obtained within the linear approximation. This suggests the perturbative solution of the cubic equation near this root:

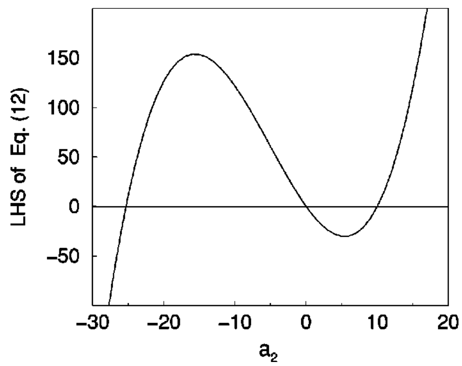

FIG. 1. The left hand side of Eq. (12) over a_2 for ϵ=0.8. Obviously Eq. (12) has three real solutions.

$$a_2 = a_2^{\rm NE}\left[1 - \frac{1005(1-\epsilon^2) - 4096T_3}{6080(1-\epsilon^2) - 4096T_2}a_2^{\rm NE} + \cdots\right], \quad (17)$$

where we do not write explicitly terms of the order $\mathcal{O}([a_2^{\rm NE}]^3)$ and higher. In Fig. 2 the dependence of $a_2^{\rm NE}$ and of the corresponding improved value a_2 are shown as a function of the restitution coefficient ϵ. As one can see from Fig. 2 the maximal deviation between these is less than 10% at small ϵ and decreases as ϵ tends to 1.

The other two roots, shown on Fig. 3 are of the order of 1 or 10, i.e., are not small. Physically, this means that one cannot cut the Sonine polynomial expansion in this case at the second term and next order terms are not negligible. Taking into account the next order terms, i.e., releasing the assumption that $a_p \simeq 0$ for $p>2$, breaks down the above analysis, since the coefficients μ_2, μ_4 occur to be dependent not only on a_2, but on a_3, a_4, . . . as well. Thus the occurrence of several roots for the a_2, found within the above approach, which satisfy the conditions required by the scaling ansatz (4) does not imply the existence of several different scaling solutions. Nevertheless such possibility may not be completely excluded. If one assumes that few scaling distributions of the velocity may realize, depending on the initial conditions at which the HC state has been prepared, a natural question arises: Whether the particular scaling solution is stable with respect to small perturbations, and what is the domain of attraction of this particular scaling solution in some parametric space.

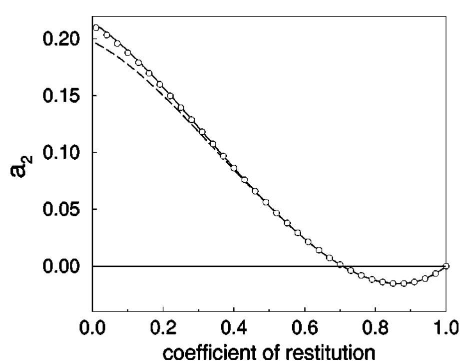

FIG. 2. The second Sonine coefficient a_2 as a function of the coefficient of restitution ϵ (full line). The dashed line shows $a_2^{\rm NE}$ in the first order approximation by van Noije and Ernst [3] according to Eq. (16). The approximation (17) is shown by circles.

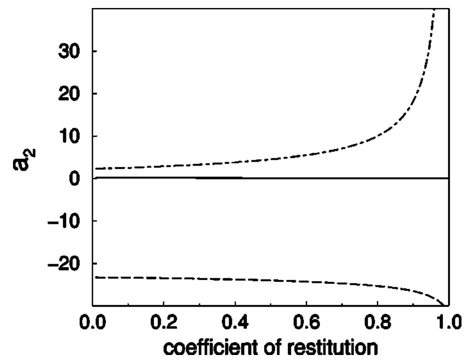

FIG. 3. The other two solutions for second Sonine coefficient a_2 of Eq. (12) over the coefficient of restitution ϵ.

Certainly, the stability problem is very complicated to be solved in general. Therefore, we restrict ourselves to the stability analysis of the scaling distribution (4) where the scaling function $\tilde{f}(\mathbf{c})$ has nonzero value of the coefficient a_2, while the other coefficients a_p with $p>2$ are negligibly small. (For this scaling solution our above results for the coefficients μ_2, μ_4 are valid). Moreover, we assume, that small perturbations of the (vanishingly small) coefficients a_p with $p>2$ do not influence the stability of the distribution, and analyze the stability only with respect to variation of the coefficient a_2.

To analyze the stability of the velocity distribution we write it in a more general form

$$f(\mathbf{v},t) = \frac{n}{v_0^3(t)}\tilde{f}(\mathbf{c},t) \quad (18)$$

which leads, as it easy to show, to the following generalization of Eq. (5) [6]:

$$\frac{\mu_2}{3}\left(3 + c_1\frac{\partial}{\partial c_1}\right)\tilde{f}(\mathbf{c},t) + B^{-1}\frac{\partial}{\partial t}\tilde{f}(\mathbf{c},t) = \tilde{I}(\tilde{f},\tilde{f}) \quad (19)$$

with the collisional integral and coefficients μ_p being now time dependent. The quantities $\langle c^p\rangle$ also depend now on time, while temperature evolves still according to Eq. (8).

Using $\tilde{f} = \phi(c)[1 + a_2(t)S_2(c^2)]$ and performing essentially the same manipulations which led before to Eq. (12), we find for the coefficient $a_2(t)$:

$$\dot{a}_2 - (4/3)B\mu_2(1+a_2) + (4/15)B\mu_4 = 0 \quad (20)$$

with μ_2, μ_4 still given by Eqs. (13), (14), but with the time-dependent coefficient $a_2(t)$. Writing the above value $B(t)$ as

$$B(t) = (8\pi)^{-1/2}\tau_c(0)^{-1}u(t)^{1/2}, \quad (21)$$

$$\tau_c(0)^{-1} \equiv 4\pi^{1/2}g_2(\sigma)\sigma^2 n T_0^{1/2}, \quad (22)$$

where $\tau_c(0)$ is related to the initial mean-collision time at the initial temperature T_0, and $u(t) \equiv T(t)/T_0$ is the reduced temperature, we recast Eq. (20) into the form

$$\frac{da_2}{d\hat{t}} = \frac{\sqrt{2/\pi}}{15}u^{1/2}F(a_2), \quad (23)$$

where $\hat{t}$ is the reduced time, measured in units of $\tau_c(0)$, and where we define a function

$$F(a_2) \equiv 5\mu_2(1+a_2) - \mu_4. \tag{24}$$

The form of the function $F(a_2)$ for some particular value of ϵ is shown on Fig. 1. This form of $F(a_2)$ persists for all physical values of the restitution coefficient, $0 \leqslant \epsilon \leqslant 1$. There are three different roots, $F(a_2^{(i)})=0$, $i=1,2,3$, which make da_2/dt vanish yielding the scaling form for the solution of the Enskog-Boltzmann equation. The stability of the scaling solution, corresponding to $a_2^{(i)}$ requires for the derivative dF/da_2, taken at $a_2^{(i)}$ to be negative, since only in this case a small deviation $a_2 - a_2^{(i)}$ from $a_2^{(i)}$, corresponding to a scaling solution will decay with time. As one can see from Fig. 1 only the middle root, which corresponds to small values of a_2, and is close to $a_2^{\rm NE}$, predicted by linear theory [3], has negative dF/da_2, and thus is stable. We also observed that for any $0 \leqslant \epsilon \leqslant 1$ the point $a_2=0$ belongs to the attractive interval of this stable root. Naturally, this means that an initial Maxwellian distribution will relax to the non-Maxwellian with $a_2 \approx a_2^{\rm NE}$.

Note that relaxation of any (small) perturbation to this value of a_2 occurs, as it follows from Eq. (23), on the collision time scale, i.e., practically ‘‘immediately’’ on the time scale which describes the evolution of the temperature. Therefore we conclude, that the scaling solution of the Enskog-Boltzmann equation with a_2 corresponding to the middle root of the function $F(a_2)$, given with a high accuracy by Eqs. (17),(16), and with negligibly small other coefficients $a_3, a_4, \ldots$, of the Sonine polynomial expansion is a stable one with respect to (relatively) small perturbations.

In conclusion, we analyzed the velocity distribution function of a granular gas with constant restitution coefficient in the regime of homogeneous cooling. We assume that the deviations from the Maxwellian distribution may be described using only the leading term in the Sonine polynomial expansion, with all other high-order terms discarded. In this approach the deviations from the Maxwellian distribution are completely characterized by the magnitude of the coefficient a_2 of the leading term. We go beyond previous linear theories and perform a complete analysis (on the level of the description chosen), without discarding any nonlinear with respect to a_2 terms.

Performing the stability analysis of the scaling solution of the Enskog-Boltzmann equation we observe that only one value of a_2, obtained within our nonlinear analysis corresponds to a stable scaling solution. We also report corrections for this value of a_2 with respect to the previous result of the linear theory. These corrections are small (less than 10%) for all values of the restitution coefficient ϵ and vanishes as ϵ tends to unity in the elastic limit.

[1] I. Goldhirsch and G. Zanetti, Phys. Rev. Lett. **70**, 1619 (1993); S. McNamara and W.R. Young, Phys. Rev. E **50**, R28 (1993); F. Spahn, U. Schwarz, and J. Kurths, Phys. Rev. Lett. **78**, 1596 (1997); T. Aspelmeier, G. Giese, and A. Zippelius, Phys. Rev. E **57**, 857 (1997); P. Deltour and J.L. Barrat, J. Phys. I **7**, 137 (1997); R. Brito and M.H. Ernst, Europhys. Lett. **43**, 497 (1998).

[2] T.P.C. van Noije and M.H. Ernst, R. Brito, and J.A.G. Orza, Phys. Rev. Lett. **79**, 411 (1997).

[3] T.P.C. van Noije and M.H. Ernst, Granular Matter **1**, 57 (1998).

[4] A. Goldshtein and M. Shapiro, J. Fluid. Mech. **282**, 75 (1995).

[5] N.V. Brilliantov, F. Spahn, J.-M. Hertzsch, and T. Pöschel, Phys. Rev. E **53**, 5382 (1996); T. Schwager and T. Pöschel, *ibid.* **57**, 650 (1998); R. Ramírez, T. Pöschel, N.V. Brilliantov, and T. Schwager, *ibid.* **60**, 4465 (1999).

[6] N.V. Brilliantov and T. Pöschel, in *Granular Gases*, edited by T. Pöschel and S. Luding (Springer, Berlin, 2000); N.V. Brilliantov and T. Pöschel, e-print cond-mat/9911212.

[7] J.J. Brey, J.W. Dufty, C.S. Kim, and A. Santos, Phys. Rev. E **58**, 4638 (1998).

[8] S.E. Esipov and T. Pöschel, J. Stat. Phys. **86**, 1385 (1997).

[9] P. Resibois and M. de Leener, *Classical Kinetic Theory of Fluids* (Wiley, New York, 1977).

[10] N.F. Carnahan and K.E. Starling, J. Chem. Phys. **51**, 635 (1969).